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Logarithmically homogeneous preferences

Mitsunobu Miyake

Graduate School of Economics and Management, Tohoku University, Sendai 980-8576, Japan

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ABSTRACT

An extended-real-valued function on $\mathbb{R}^n_+$ is called logarithmically homogeneous if it is given by the logarithmic transformation of a homogeneous function on $\mathbb{R}^n_+$. Specifying a consumer's preference on the consumption set by a difference comparison relation, this paper provides some axioms on the relation under which the full class of utility functions representing the relation are logarithmically homogeneous. It is also shown that all the utility functions are strongly concave and all the indirect utility functions are logarithmically homogeneous. Moreover, the additively separable logarithmic utility functions are derived by strengthening one of the axioms.

1. Introduction

As a utility function exhibiting the *Marshallian constancy*, that is, the marginal utility of income depends only on the income levels, Samuelson (1942, Equation (41)) introduces the following form of utility function[1]: a utility function U defined on the consumption set $X = \mathbb{R}^n_+$ is *logarithmically homogeneous* if and only if there is a 1-homogeneous function[2] u on X and two parameters $a > 0$ and b such that $U(\mathbf{x}) = a \cdot \log u(\mathbf{x}) + b$ for all $\mathbf{x} \in X$.[3]

All the logarithmically homogeneous utility functions U are *strongly concave* on X under the strict quasi-convexity assumption for the underlying preference orderings and the demand functions derived from the logarithmically homogeneous utility functions are 1-homogeneous with respect to prices. Therefore, if we assume the logarithmically homogeneous utility functions for the market models in an applied welfare analysis, we can derive some normative conclusions (such as equity-regarding policy prescriptions) based on the strong concavity, while the market models with the homogeneous demand functions are simple and computationally tractable. In particular, an *additively separable logarithmic utility function* $U^*(\mathbf{x}) = a_1 \cdot \log x_1 + \cdots + a_n \cdot \log x_n + b$ is a logarithmically homogeneous utility function whose (underlying) 1-homogeneous function coincides with a Cobb–Douglas function, and then the competitive equilibria of the market models with the additively separable logarithmic utilities can be computed using the corresponding Cobb–Douglas demand functions, $d(\mathbf{p}, I) = (a_1 I/p_1, \ldots, a_n I/p_n)$.

In the (neoclasical) ordinal utility theory, a utility function is defined as a numerical indicator representing a consumer's preference ordering to explain the consumer's demand behavior in the market model, and the utility function is unique up to the monotone transformations. A characterization theorem of the representability of a preference ordering by the 1-homogeneous utility function is given by Katzner (1970, Theorem 2.3-2), who shows that the preference ordering satisfies the monotonicity, continuity and homotheticity axioms if and only if there exists at least one 1-homogeneous utility function representing the preference ordering.[4] Since the logarithmic function is monotone, it follows from Katzner's characterization result that the three axioms above are necessary and sufficient for a preference

E-mail address: miyake@econ.tohoku.ac.jp.

1 See also Katzner (1967) and Rader (1976). Mantel (1976) and Mas-Colell (1985, page 197) assume the utility functions to prove the Sonnenschein–Mantel–Debreu theorem in the competitive market model.

2 For a given integer r, a real-valued function f on X is called r-homogeneous if and only if f is homogeneous of degree r, i.e., $f(\lambda \cdot \mathbf{x}) = \lambda^r \cdot f(\mathbf{x})$ for all $\mathbf{x} \in X$ and all $\lambda > 0$.

3 Since a (typical) indirect utility function corresponding to the logarithmically homogeneous utility function can be written as $V(\mathbf{p}, I) = \alpha \cdot \log[I/v(\mathbf{p})] + \beta$, where v is a (-1)-homogeneous function and a $\alpha > 0$ and β are parameters, it holds that $\partial V(\mathbf{p}, I)/\partial I = \alpha \cdot [v(\mathbf{p})/I] \cdot [1/v(\mathbf{p})] = \alpha/I$, which is the Marshallian constancy.

4 See also Dow and Werlang (1992), Candeal and Induráin (1995) and Bosi et al. (2000).

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ordering to be represented by a logarithmically homogeneous utility function. However, this characterization result ensures not only the existence of a logarithmically homogeneous utility function but also the existence of some utility functions that are not logarithmically homogeneous, because the logarithmic homogeneity of the utility function is not assured for some monotone transformations, that is, the logarithmic homogeneity is a cardinal property of the (ordinally determined) utility function.[5] As a general rule, a normative conclusion should not be derived from a specific property of utility function that depends on the selection of utility function in the welfare/normative analysis, and then the preference foundation of the utility functions for the analysis is not well-established by the characterization result above.[6]

To avoid this phenomenon that is intrinsic in the ordinal utility theory, this paper adapts the cardinal utility theory based on the difference (intensity) comparisons developed by Alt (1936), Shapley (1975) and others, where a utility function is an indicator representing a consumer's absolute welfare levels (in a classical sense) and is unique up to the positive affine transformations.[7] In particular, Shapley (1975) shows that a difference comparison relation on the positive real-line, R_{++} satisfies the monotonicity, continuity, consistency and crossover axioms if and only if there is a cardinal utility function representing the relation. Miyake (2014) introduces a new axiom called the homogeneity axiom for the difference comparison relation on R_{++} and shows that the logarithmic utility function $u(x) = \log x$ can be characterized by three axioms: monotonicity, consistency and homogeneity axioms, where the former two axioms are introduced by Shapley (1975).

This paper attempts to derive the logarithmically homogeneous utility function by extending the characterization results above to the general case where the difference comparison relations are defined on the n-dimensional, non-negative Euclidean space $\mathbb{R}^n_+$. Concretely, this paper provides a new homogeneity axiom for the difference comparison relation on $\mathbb{R}^n_+$, which is stronger than not only Katzner's (1970) homotheticity axiom but also Miyake's (2014) homogeneity axiom. Hence there exists a 1-homogeneous utility function $u(\mathbf{x})$ representing some parts of the relation on $\mathbb{R}^n_+$, and $U(\mathbf{x}) = \log u(\mathbf{x})$ is a logarithmically homogeneous utility function representing the relation on the ray $\{x \in \mathbb{R}^n_+ : x_1 = x_2 = \cdots = x_n\}$. Using these facts, we show that the full class of utility functions representing a difference comparison relation on $\mathbb{R}^n_+$ are logarithmically homogeneous if and only if the relation satisfies the four mutually independent axioms: monotonicity, continuity, consistency and homogeneity axioms, where the former three axioms are given by re-stating Shapley's (1975) corresponding axioms on $\mathbb{R}^n_+$, respectively.[8]

Moreover, under some axioms (including the quasi-convexity axiom) for ensuring the well definedness of a demand function and indirect utility function, it is shown that the homogeneity axiom is necessary and sufficient for all the indirect utility functions to be logarithmically homogeneous, and that the additively separable logarithmic utility functions are derived by strengthening the homogeneity axiom only in the axiomatic characterizations above.[9]

The characterization results clarify the scope of applicability of logarithmically homogeneous utility functions. In particular, since all the logarithmically homogeneous utility functions above are strongly concave under the quasi-convexity axiom, we can assume the logarithmic homogeneity condition and the strong concavity condition for the consumers' utility functions simultaneously in the market model, independent of the selections of utility functions.

The next section introduces the basic definitions and axiomatically derives the logarithmically homogeneous utility function. Sections 3 and 4 derive the logarithmically homogeneous indirect utility function and the additively separable logarithmic utility function, respectively.

2. Logarithmically homogeneous utility functions

The consumption set X is defined by $X = \mathbb{R}^n_+$. A difference on X is a transition (path) from a vector $\mathbf{x} \in X$ to a vector $\mathbf{y} \in X$, and the difference from $\mathbf{x}$ to $\mathbf{y}$ is denoted by an ordered pair $(\mathbf{x}, \mathbf{y})$. The set of all admissible differences is defined by $Y = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \gg \mathbf{0}\}$,[10] where $\mathbf{0}$ is the origin of X. We identify Y with a subset of the Euclidean space $R^n_{++} \times \mathbb{R}^n_+ \subset R^{2n}$, and Y is endowed with the (relative) Euclidean metric topology. A *difference (comparison) relation* on X is a complete and transitive binary relation on Y. The expression $(\mathbf{x}, \mathbf{y}) \succsim (\mathbf{z}, \mathbf{w})$ means that the transition from $\mathbf{x}$ to $\mathbf{y}$ is preferred to the transition from $\mathbf{z}$ to $\mathbf{w}$. The symmetric and asymmetric parts of $\succsim$ are denoted by $\sim$ and $\succ$, respectively. A function $U : X \to \mathbb{R} \cup \{-\infty\}$ is called a utility function representing a difference relation $\succsim$ if and only if

$$(\mathbf{x}, \mathbf{y}) \succsim (\mathbf{z}, \mathbf{w}) \Leftrightarrow U(\mathbf{y}) - U(\mathbf{x}) \geq U(\mathbf{w}) - U(\mathbf{z}) \quad \text{for all } (\mathbf{x}, \mathbf{y}), (\mathbf{z}, \mathbf{w}) \in Y.^{11} \tag{1}$$

A utility function U representing $\succsim$ is *logarithmically homogeneous* if and only if

U is continuously increasing on X, and

$$U(\lambda \cdot \mathbf{x}) = \theta \cdot \log \lambda + U(\mathbf{x}) \quad \text{for all } \lambda > 0 \text{ and all } \mathbf{x} \in X \text{ with } \mathbf{x} \neq \mathbf{0}, \tag{2}$$

where $\theta = U(e, \ldots, e) - U(1, \ldots, 1) > 0$.[12] Another form of the definition can be stated as follows: a utility function U representing

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5 For the computation of a competitive equilibrium, the ordinal characterization result can be used towards the justification of logarithmically homogeneous utility functions, since a competitive equilibrium is determined independent of the selection of the utility functions.

6 In case of uncertainty, Kaneko (1984, Appendix), Wakker and Zank (1999) and Köbberling and Wakker (2003) derive the cardinal expected utility functions representing the intensity comparisons so that all the utility functions satisfy the expected utility hypothesis, which is crucial for relaxing the common knowledge assumptions of the non-cooperative bargaining games.

7 For a survey of the cardinal utility theory, see Köbberling (2006) and Hudík (2014). The growth rate comparison for the transitions of GDP can be recognized as a typical difference comparison. For example, if the t-year's annual growth rate ($\equiv [\mathrm{GDP}(t) - \mathrm{GDP}(t-1)]/\mathrm{GDP}(t-1)$) is greater than t^*-year's growth rate, then the transition from $\mathrm{GDP}(t-1)$ to $\mathrm{GDP}(t)$ is preferred to the transition from $\mathrm{GDP}(t^*-1)$ to $\mathrm{GDP}(t^*)$. For the growth rate comparison, see Mandelbrot (1960, Section 1.5), Graff (2014) and Miyake (2014). However, Kaneko (1984) points out a problem with the method of eliciting the relation behaviorally. For such a methodological problem, see Köbberling and Wakker (2003, Section 3) and Qin and Shubik (2015).

8 This result implies that the crossover axiom is redundant in the axiomatization for the logarithmically homogeneous utility function.

9 The strengthened homogeneity axiom is stronger than the strong homotheticity axiom (budget-invariance axiom) introduced by Trockel (1989) to characterize the Cobb–Douglas preferences.

10 Since the value of $(-\infty) - (-\infty)$ is left undefined in the extended real number system, we cannot evaluate $(\mathbf{0}, \mathbf{0})$ numerically, and then this restriction on the domain of $\succsim$ is needed. See Royden and Fitzpatrick (2010, Section 1.1) for the extended real numbers.

11 As the arithmetic rules for the extended real numbers, we assume that $a - (-\infty) = -(-\infty) > b$ and $(-\infty) - a = (-\infty) < b$ for all $a, b \in R$.

12 Condition (2) implies that $U(\mathbf{0}) = \lim_{\lambda \to +0} U(\lambda, \ldots, \lambda) = \lim_{\lambda \to +0} [\theta \cdot \log \lambda + U(1, \ldots, 1)] = -\infty$. Setting $\mathbf{x} = (1, \ldots, 1)$ and $\lambda = e$ in (2), we have $U(e, \ldots, e) = \theta + U(1, \ldots, 1)$, which implies $\theta = U(e, \ldots, e) - U(1, \ldots, 1)$. This form of definition is standard in the mathematical programming theory. See Nesterov and Nemirovskii (1994). The case of decreasing function is considered in Section 3 as a form of function dual to (2).

$\succsim$ is *logarithmically homogeneous* if and only if

there is a continuously increasing and 1-homogeneous function

$u : X \to \mathbb{R}_+$ and a real number μ such that

$$U(\mathbf{x}) = \theta \cdot \log u(\mathbf{x}) + \mu \text{ for all } \mathbf{x} \in X, \tag{3}$$

where $\theta = U(e, \ldots, e) - U(1, \ldots, 1) > 0$.[13] The two definitions (2) and (3) are equivalent. In fact, we have the following lemma:

Lemma 1. *A utility function U representing $\succsim$ satisfies (2) if and only if U satisfies (3).*

Lemma 1 is proved in Appendix B. In order to characterize the logarithmically homogeneous utility function, we introduce the axioms:

Monotonicity: $\mathbf{x} \gg \mathbf{y} \Rightarrow (\mathbf{z}, \mathbf{x}) \succ (\mathbf{z}, \mathbf{y})$ for all $\mathbf{z} \gg \mathbf{0}$.

Continuity: $\{(\mathbf{x}, \mathbf{y}) \in Y : (\mathbf{x}, \mathbf{y}) \succsim (\mathbf{z}, \mathbf{w})\}$ and $\{(\mathbf{x}, \mathbf{y}) \in Y : (\mathbf{z}, \mathbf{w}) \succsim (\mathbf{x}, \mathbf{y})\}$ are (relatively) closed in Y for all $(\mathbf{z}, \mathbf{w}) \in Y$.

Consistency: If $(\mathbf{z}, \mathbf{x}) \succsim (\mathbf{z}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \gg \mathbf{0}$, then $(\mathbf{y}, \mathbf{z}) \succsim (\mathbf{x}, \mathbf{z})$ and $(\mathbf{w}, \mathbf{x}) \succsim (\mathbf{w}, \mathbf{y})$ for all $\mathbf{w} \gg \mathbf{0}$.

Homogeneity: $(\mathbf{x}, \mathbf{y}) \sim (\lambda \cdot \mathbf{x}, \lambda \cdot \mathbf{y})$ for all $(\mathbf{x}, \mathbf{y}) \in Y$ and all $\lambda > 0$.

The monotonicity and continuity axioms are standard. The consistency axiom enables us to derive a binary relation from $\succsim$, which can be interpreted as a standard preference ordering. Formally, for a given difference relation $\succsim$, a binary relation $\succsim'$ on X induced from $\succsim$ is defined by

$$\mathbf{x} \succsim' \mathbf{y} \Leftrightarrow (\mathbf{z}, \mathbf{x}) \succsim (\mathbf{z}, \mathbf{y}) \quad \text{for some } \mathbf{z} \gg \mathbf{0} \tag{4}$$

for $\mathbf{x}, \mathbf{y} \in X$. The expression $\mathbf{x} \succsim' \mathbf{y}$ means that $\mathbf{x}$ is preferred to $\mathbf{y}$, and the binary relation $\succsim'$ is called the *level (comparison) relation* of $\succsim$. We have the following lemma:

Lemma 2. *Suppose that a difference relation $\succsim$ satisfies the monotonicity, continuity and consistency axioms. Then the following assertions hold:*

(i) *$\succsim$ is Archimedean, that is, $(\mathbf{z}, \mathbf{x}) \succ (\mathbf{z}, \mathbf{y})$ and $\mathbf{w} \gg \mathbf{0} \Rightarrow (\mathbf{z}, \mathbf{x}) \sim (\mathbf{z}, \mathbf{y} + \lambda \cdot \mathbf{w})$ for some $\lambda > 0$.*
(ii) *$(\mathbf{z}, \mathbf{x}) \succsim (\mathbf{z}, \mathbf{y}) \Rightarrow (\mathbf{w}, \mathbf{x}) \succsim (\mathbf{w}, \mathbf{y})$ for all $\mathbf{w} \gg \mathbf{0}$.*[14]
(iii) *$\succsim'$ is complete, transitive, monotone and continuous on X.*
(iv) *$\mathbf{x} \sim' \mathbf{y} \Rightarrow (\mathbf{z}, \mathbf{x}) \sim (\mathbf{z}, \mathbf{y})$ for all $\mathbf{z} \gg \mathbf{0}$, and $\mathbf{x} \sim' \mathbf{y} \Rightarrow (\mathbf{z}, \mathbf{x}) \succ (\mathbf{z}, \mathbf{y})$ for all $\mathbf{z} \gg \mathbf{0}$, where $\sim'$ and $\succ'$ are the symmetric and asymmetric parts of $\succsim'$, respectively.*
(v) *$\succsim'$ is Archimedean, that is, $\mathbf{x} \succ' \mathbf{y}$ and $\mathbf{z} \gg \mathbf{0} \Rightarrow \mathbf{x} \sim' \mathbf{y} + \lambda \cdot \mathbf{z}$ for some $\lambda > 0$.*
(vi) *If $\mathbf{x}_i \sim' \mathbf{y}_i$ for $i = 1, 2, 3, 4$ with $\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_3, \mathbf{y}_3 \gg \mathbf{0}$, then $(\mathbf{x}_1, \mathbf{x}_2) \succsim (\mathbf{x}_3, \mathbf{x}_4) \Leftrightarrow (\mathbf{y}_1, \mathbf{y}_2) \succsim (\mathbf{y}_3, \mathbf{y}_4)$.*
(vii) *$\mathbf{x} \succsim' \mathbf{0}$ for all $\mathbf{x} \in \mathbf{X}$.*

Lemma 2(ii) implies that if a difference relation $\succsim$ satisfies the three axioms, the underlying level relation $\succsim'$ of $\succsim$ is well-defined, independent of the selection of the reference point $\mathbf{z} \in X$ in (4). Lemma 2 is proved in Appendix B.

The homogeneity axiom above requires that if two differences are indifferent, then the indifference is invariant against the changes of the unit of consumption goods that are common for all the goods. Under the three axioms, the homogeneity axiom implies Katzner's (1970, Theorem 2.3-2) homotheticity condition, which requires the indifference relation $\sim'$ to be scale-invariant. Further Katzner proves the existence of a 1-homogeneous function representing the level relation.[15] Formally, we have the following lemma:

Lemma 3. *Suppose that a difference relation $\succsim$ satisfies the monotonicity, continuity, consistency and homogeneity axioms. Then the following assertions hold:*

(i) $\succsim'$ is homothetic, that is, $\mathbf{x} \sim' \mathbf{y} \Rightarrow \lambda \cdot \mathbf{x} \sim' \lambda \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in X$ and all $\lambda > 0$.[16]
(ii) $(\alpha \cdot \mathbf{e}, \beta \cdot \mathbf{e}) \succsim (\gamma \cdot \mathbf{e}, \delta \cdot \mathbf{e}) \Leftrightarrow \beta/\alpha = \delta/\gamma$ for all $a, \beta, \gamma, \delta \in \mathbb{R}_+$ with $\alpha, \gamma > 0$, where $\mathbf{e} \equiv (1, 1, \ldots, 1)$.
(iii) There is a continuously increasing function $u^*: X \to \mathbb{R}_+$ such that u^* is 1-homogeneous and that $\mathbf{x} \succsim' \mathbf{y} \Leftrightarrow u^*(\mathbf{x}) \geq u^*(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$.[17]
(iv) A function $u : X \to \mathbb{R}$ satisfies all the conditions in Lemma 3(iii) if and only if there exists $\alpha > 0$ such that $u(\mathbf{x}) = \alpha \cdot u^*(\mathbf{x})$ for all $\mathbf{x} \in X$.

The function u^* in Lemma 3(iii) is called the 1-homogeneous function representing $\succsim'$. The proof of Lemma 3 is given in Appendix B.

As shown in Fig. 1, the homotheticity condition requires that $\mathbf{a} \sim' \mathbf{b} \Rightarrow 2\mathbf{a} \sim' 2\mathbf{b}$, that is, indifference curves are proportionally expanding, and the homogeneity axiom requires not only this proportionality property, but also that any two parallel transitions between the two expansion paths are indifferent such as $(\mathbf{a}, 2\mathbf{b}) \sim (2\mathbf{a}, 4\mathbf{b})$, by which we can show, using (1) together, that $U(2\mathbf{b}) - U(\mathbf{a}) = U(4\mathbf{b}) - U(2\mathbf{a})$, that is, the utility differences are invariable for the two transitions.

It is obvious that the four axioms above hold if there is a logarithmically homogeneous utility function. The following theorem implies that the four axioms are sufficient for the existence of the logarithmically homogeneous utility function:

Theorem 1. (A) *The following three statements are mutually equivalent:*

(i) *A difference relation $\succsim$ satisfies the monotonicity, continuity, consistency and homogeneity axioms.*
(ii) *A difference relation $\succsim$ is represented by a logarithmically homogeneous utility function.*
(iii) *There is a utility function representing $\succsim$, and all the utility functions representing $\succsim$ are logarithmically homogeneous. Moreover, the utility function representing $\succsim$ is unique up to the positive affine transformations.*

(B) *Suppose that a difference relation $\succsim$ satisfies not only all of the axioms in Assertion (Ai) above but also the quasi-convexity axiom:*

Quasi-convexity *(quasi-convexity of $\succsim'$): If $(\mathbf{x}, \mathbf{y}) \succsim (\mathbf{x}, \mathbf{z})$, $(\mathbf{x}, \mathbf{y}^*) \succsim (\mathbf{x}, \mathbf{z})$, $\mathbf{y} \neq \mathbf{y}^*$ and $\mathbf{z} \gg \mathbf{0}$, then*
$(\mathbf{x}, \tau \cdot \mathbf{y} + (1 - \tau) \cdot \mathbf{y}^*) \succ (\mathbf{x}, \mathbf{z})$ for all $\tau \in (0, 1)$.[18]

Then all the utility functions representing $\succsim$ are strongly concave *as well as logarithmically homogeneous.*

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13 It holds by (3) that $U(e, \ldots, e) = \theta \cdot \log u(e, \ldots, e) + b$, and that $U(1, \ldots, 1) = \theta \cdot \log u(1, \ldots, 1) + b$. Hence we have $U(e, \ldots, e) - U(1, \ldots, 1) = \theta \cdot \log u(e, \ldots, e)/u(1, \ldots, 1) = \theta \cdot \log e = \theta$. This form of definition is introduced by Samuelson (1942) and Katzner (1967).

14 Lemma 2(ii) is suggested by the referee.

15 See also Dow and Werlang (1992, Proposition 1.5 and Theorem 1.7), Candeal and Induráin (1995, Section 4) and Bosi et al. (2000).

16 The difference relations derived from the Cobb–Douglas utility functions do not satisfy our homogeneity axiom, but the Cobb–Douglas level relations satisfy Katznerś homotheticity axiom. In fact, if $U(x_1, x_2) = (x_1 x_2)^{1/2}$, it holds that $U(y_1, y_2) - U(x_1, x_2) = (y_1 y_2)^{1/2} - (x_1 x_2)^{1/2} < [(\lambda y_1 \lambda y_2)^{1/2} - (\lambda x_1 \lambda x_2)^{1/2}] = U(\lambda y_1, \lambda y_2) - U(\lambda x_1, \lambda x_2)$ for all $(y_1, y_2) \gg (x_1, x_2)$ and all $\lambda > 1$, and that $(y_1 y_2)^{1/2} = (x_1 x_2)^{1/2} \Rightarrow (\lambda y_1 \lambda y_2)^{1/2} = (\lambda x_1 \lambda x_2)^{1/2}$ for all $\lambda > 0$.

17 Since u^* is 1-homogeneous, it holds that $u^*(2 \cdot \mathbf{0}) = 2 \cdot u^*(\mathbf{0})$, which implies $u^*(\mathbf{0}) = 0$. The *money-metric* utility functions introduced by Weymark (1985) satisfy all the conditions in Lemma 3(iii).

18 Although the quasi-convexity axiom introduced here is so weak that all the Cobb–Douglas preference orderings satisfy the axiom, it is sufficient for ensuring the uniqueness of the demand vector for each price vector.

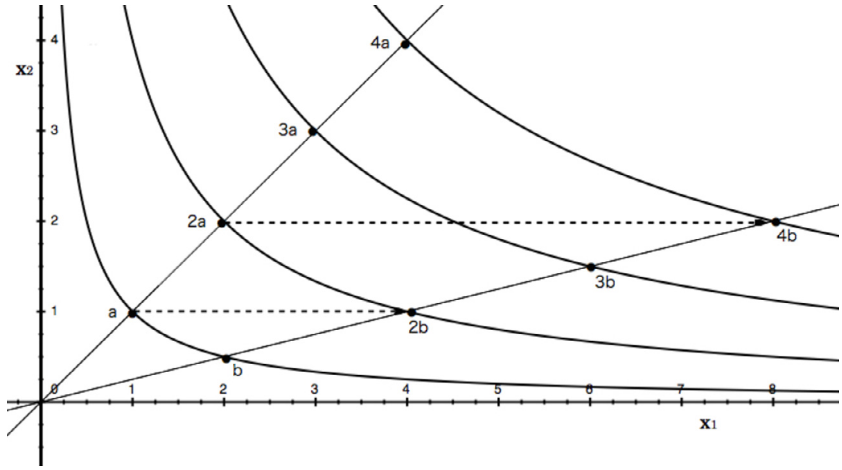

Fig. 1. Indifference map of the level relation.

Theorem 1 is proved in Appendix A. Theorem 1(A) implies that the four axioms are necessary and sufficient for the existence of a logarithmically homogeneous utility function and for all the utility functions to be logarithmically homogeneous. Moreover, under all the axioms in Theorem 1(Ai), Theorem 1(A) and (3) together imply that an extended-real-valued function U on X is a utility function representing $\succsim$ if and only if there exists $\alpha > 0$ and β such that $U(\mathbf{x}) = \alpha \cdot \log u(\mathbf{x}) + \beta$ for all $\mathbf{x} \in X$, where $u(\mathbf{x})$ is a 1-homogeneous function representing $\succsim'$ given by Lemma 3(iii). In particular, it holds by Lemma 3(iv) that the formula: $U(\mathbf{x}) = \alpha \cdot \log u(\mathbf{x}) + \beta$ is well-defined, independent of the selection of u.

All the axioms in Theorem 1(Ai) are mutually independent, which can be shown by constructing the counter examples. Define a difference relation $\succsim_1$ by

$$(\mathbf{x}, \mathbf{y}) \succsim_1 (\mathbf{z}, \mathbf{w}) \quad \text{for all } (\mathbf{x}, \mathbf{y}), (\mathbf{z}, \mathbf{w}) \in Y.$$

The difference relation $\succsim_1$ satisfies the continuity, consistency and homogeneity axioms, but it does not satisfy the monotonicity axiom, which implies that the monotonicity axiom is independent of the other axioms. Define a difference relation $\succsim_2$ by

$$(\mathbf{x}, \mathbf{y}) \succsim_2 (\mathbf{z}, \mathbf{w}) \Leftrightarrow (y_1/x_1, y_2/x_2, \ldots, y_n/x_n) \succsim_L' (w_1/z_1, w_2/z_2, \ldots, w_n/z_n) \quad \text{for all } (\mathbf{x}, \mathbf{y}), (\mathbf{z}, \mathbf{w}) \in Y,$$

where $\succsim_L{}'$ is the lexicographic ordering on $\mathbb{R}^n_+$.[19] Although the difference relation $\succsim_2$ satisfies the monotonicity, consistency and homogeneity axioms, it does not satisfy the continuity axiom, which implies that the continuity axiom is independent. Define a difference relation $\succsim_3$ by

$$(\mathbf{x}, \mathbf{y}) \succsim_3 (\mathbf{z}, \mathbf{w}) \Leftrightarrow \Sigma_i (y_i/x_i) \log[1 + (y_i/x_i)] \geq \Sigma_i (w_i/z_i) \log[1 + (w_i/z_i)] \quad \text{for all } (\mathbf{x}, \mathbf{y}), (\mathbf{z}, \mathbf{w}) \in Y.$$

Then $\succsim_3$ satisfies the monotonicity, continuity and homogeneity axioms, but it does not satisfy the consistency axiom,[20] which implies that the consistency axiom is independent. Define a difference relation $\succsim_4$ by

$$(\mathbf{x}, \mathbf{y}) \succsim_4 (\mathbf{z}, \mathbf{w}) \Leftrightarrow \prod_i y_i - \prod_i x_i \geq \prod_i w_i - \prod_i z_i \quad \text{for all } (\mathbf{x}, \mathbf{y}), (\mathbf{z}, \mathbf{w}) \in Y.$$

Then $\succsim_4$ satisfies the monotonicity, continuity and consistency axioms, but it does not satisfy the homogeneity axiom, which implies that the homogeneity axiom is independent.

3. Logarithmically homogeneous indirect utility functions

In the previous section, an extended-real-valued function U on X is defined to be a utility function representing a difference relation $\succsim$ if and only if U is order-preserving with respect to the difference relation $\succsim$, that is, U satisfies (1) in Section 2. Accordingly, this section defines the indirect utility function V derived from the difference relation $\succsim$, assuming some axioms on to ensure the well-definedness of the indirect utility function, and shows that the homogeneity axiom is necessary and sufficient for all the indirect utility functions to be logarithmically homogeneous.

Let $\succsim$ be a difference relation on $X = \mathbb{R}^n_+$, and let $\succsim'$ be the level relation of $\succsim$. We assume that $\succsim$ satisfies the monotonicity, continuity, consistency and quasi-convexity axioms on X, which implies that $\succsim'$ satisfies the monotonicity, continuity and quasi-convexity on X. Setting $\mathbb{P} \equiv \mathbb{R}^n_{++}$ and $\mathbb{I} \equiv \mathbb{R}^1_{++}$, the *demand function* $d : \mathbb{P} \times \mathbb{I} \to X$ can be defined by the unique element of $\{\mathbf{x} \in X : \mathbf{px} \leq I \text{ and } \mathbf{x} \succsim' \mathbf{y} \text{ for all } \mathbf{y} \in X \text{ with } \mathbf{py} \leq I\}$ for each $(\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}$, where $\mathbf{px} \equiv p_1 \cdot x_1 + p_2 \cdot x_2 + \cdots + p_n \cdot x_n$. A function $V : \mathbb{P} \times \mathbb{I} \to \mathbb{R} \cup \{-\infty\}$ is called an *indirect* utility function of $\succsim$ if and only if

$$(d(\mathbf{p}, I), d(\mathbf{q}, J)) \succsim (d(\mathbf{p}^*, I^*), d(\mathbf{q}^*, J^*)) \Leftrightarrow V(\mathbf{q}, J) - V(\mathbf{p}, I) \geq V(\mathbf{q}^*, J^*) - V(\mathbf{p}^*, I^*)$$

for all $(\mathbf{p}, I), (\mathbf{q}, J), (\mathbf{p}^*, I^*), (\mathbf{q}^*, J^*) \in \mathbb{P} \times \mathbb{I}$ with $(d(\mathbf{p}, I), d(\mathbf{q}, J)), (d(\mathbf{p}^*, I^*), d(\mathbf{q}^*, J^*)) \in Y$.

In particular, an indirect utility function V^* of $\succsim$ is called *logarithmically homogeneous* (*with respect to price vectors*) if and only if

$$\begin{aligned} &\text{there is a continuously decreasing and } (-1)\text{-homogeneous} \\ &\quad \text{function } v^* : \mathbb{R}^n_{++} \cup \{+\infty\} \to \mathbb{R}_+ \\ &\quad \text{with } v^*(+\infty) = 0 \text{ and two real numbers } \alpha > 0 \text{ and } \beta \\ &\quad \text{such that } V^*(\mathbf{p}, I) = \alpha \cdot \log v^*(p_1/I, \ldots, p_n/I) + \beta \\ &\quad \text{for all } (\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}.^{21} \end{aligned} \tag{5}$$

[19] See Mas-Colell et al. (1995, Example 3.C.1, p.46) for the lexicographic ordering.

[20] Since $((2,1,1,\ldots,1),(2,3,1,\ldots,1)) \succ_3 ((2,1,1,\ldots,1),(3,2,1,\ldots,1))$ and $((1,2,1,\ldots,1),(2,3,1,\ldots,1)) \succ_3 ((1,2,1,\ldots,1),(3,2,1,\ldots,1))$, $\succsim_3$ does not satisfy the consistency axiom.

[21] The notation $+\infty$ means $(+\infty, \ldots, +\infty)$. The budget set at $(\mathbf{p}, 0) \in \mathbb{P} \times \mathbb{I}$ coincides with $\{\mathbf{0}\}$ and $d(\mathbf{p}, 0) = \mathbf{0}$, which implies that $V^*(\mathbf{p}, 0) = \alpha \cdot \log v^*(+\infty) + \beta = -\infty$ for all $\mathbf{p} \in \mathbb{P}$.

In fact, if an indirect utility function V^* of $\succsim$ is logarithmically homogeneous, it holds that $V^*(\lambda \cdot \mathbf{p}, I) = -\alpha \cdot \log \lambda + V^*(\mathbf{p}, I)$ for all $(\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}$ and all $\lambda > 0$, which means that the definition of the logarithmic homogeneity for the indirect utility functions is consistent to the definition for the direct utility functions given by (2). The main result of this section is the following theorem:

Theorem 2. *Let $\succsim$ be a difference relation on X satisfying the monotonicity, continuity, consistency and quasi-convexity axioms on X. The following statements are mutually equivalent:*

(i) *The difference relation $\succsim$ satisfies the homogeneity axiom on X.*
(ii) *There is a logarithmically homogeneous indirect utility function of $\succsim$.*
(iii) *There is an indirect utility function of $\succsim$, and all the indirect utility functions of $\succsim$, are logarithmically homogeneous on $\mathbb{P} \times \mathbb{I}$. Moreover, the indirect utility function of $\succsim$ is unique up to the positive affine transformations.*

Theorem 2 is proved in Appendix A. Theorem 2 implies that the homogeneity axiom is necessary and sufficient for the existence of a logarithmically homogeneous indirect utility function and for all the indirect utility functions to be logarithmically homogeneous under the monotonicity, continuity, consistency and quasi-convexity axioms on X. Thus, we can assume a logarithmically homogeneous indirect utility function as a cardinal function for the consumer whose difference relation satisfies all the axioms above. Under all the axioms in Theorem 2, Theorem 2 and (5) together imply that an extended-real-valued function V on $\mathbb{P} \times \mathbb{I}$ is an indirect utility function of $\succsim$ if and only if there exists $\alpha > 0$ and β such that $V(\mathbf{x}) = \alpha \cdot \log u(d(\mathbf{p}, I)) + \beta$ for all $(\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}$, where u is a 1-homogeneous function on X satisfying all the conditions in Lemma 3(iii). In particular, setting $\alpha = 1$ and $\beta = 0$, a typical indirect utility function $V^*(\mathbf{p}, I)$ of $\succsim$ can be written as

$$V^*(\mathbf{p}, I) = \log I + \log u(d(\mathbf{p}, 1)) \quad \text{for all } (\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}, \tag{6}$$

which means that the consumer's cardinal utility values of $V^*(\mathbf{p}, I)$ can be decomposed into the income part, $\log I$ and price part, $\log u(d(\mathbf{p}, 1))$. Alternatively, setting $c(\mathbf{p}) = 1/u(d(\mathbf{p}, 1))$ in (6), another typical indirect utility function $V^0(\mathbf{p}, I)$ of $\succsim$ can be written as

$$V^0(\mathbf{p}, I) = \log[I/c(\mathbf{p})] \text{ for all } (\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}.$$

For a consumer satisfying all the axioms above, $I/c(\mathbf{p})$ and $c(\mathbf{p})$ can be recognized as a normalized income level and a (personalized) deflator, respectively, since $V^0(\mathbf{p}, I)$ is a cardinal indirect utility function of the consumer. When the price vector is fixed, the indirect utility function V^0 can be recognized as the logarithmic utility function on income levels as considered by Mandelbrot (1960, Section 1.5), Graff (2014) and Miyake (2014).

4. Additively separable logarithmic utility functions

As the utility functions not only exhibit the Marshallian constancy, but also satisfy the additive separability, Samuelson (1942, Equation 34) introduces the following form of function: a function $f : X \to \mathbb{R} \cup \{-\infty\}$ is *additively separable logarithmic* if and only if there are $n+1$ real numbers $a_1 > 0, a_2 > 0, \ldots, a_n > 0$ and b such that

$$f(\mathbf{x}) = a_1 \cdot \log x_1 + a_2 \cdot \log x_2 + \cdots + a_n \cdot \log x_n + b \quad \text{for all } \mathbf{x} \in X.$$

If a utility function $U : X \to \mathbb{R} \cup \{-\infty\}$ representing a preference $\succsim$ on X is additively separable logarithmic, that is, there are $n+1$ real numbers $a_1 > 0, a_2 > 0, \ldots, a_n > 0$ and b such that

$$U(\mathbf{x}) = a_1 \cdot \log x_1 + a_2 \cdot \log x_2 + \cdots + a_n \cdot \log x_n + b \quad \text{for all } \mathbf{x} \in X,$$

then U is called an *additively separable logarithmic utility function*. An indirect utility function $V : \mathbb{P} \times \mathbb{I} \to \mathbb{R} \cup \{-\infty\}$ of $\succsim$ is called *additively separable logarithmic* on $\mathbb{P} \times \mathbb{I}$ if there are $n+2$ real numbers $\alpha_0 > 0, \alpha_1 < 0, \alpha_2 < 0, \ldots, \alpha_n < 0$ $(\Sigma\alpha_i = -\alpha_0)$ and β such that

$$V(\mathbf{p}, I) = \alpha_0 \cdot \log I + \alpha_1 \cdot \log p_1 + \alpha_2 \cdot \log p_2 + \cdots + \alpha_n \cdot \log p_n + \beta \quad \text{for all } (\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}.\text{[22]}$$

An additively separable logarithmic indirect utility function is logarithmically homogeneous. In fact, by setting $v(t_1, \ldots, t_n) = t_1^{\alpha_1} \cdot t_2^{\alpha_2} \ldots t_n^{\alpha_n}$ for all $(t_1, \ldots, t_n) \in \mathbb{R}^n_{++}$ and $v(+\infty) = 0$, the function $v : \mathbb{R}^n_{++} \cup \{+\infty\} \to \mathbb{R}_+$ is continuously decreasing and (-1)-homogeneous on $\mathbb{R}^n_{++} \cup \{+\infty\}$, and it holds that $V(\mathbf{p}, I) = 1 \cdot \log v(p_1/I, \ldots, p_n/I) + \beta$ for all $(\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}$, which means that $V(\mathbf{p}, I)$ satisfies (5).

This section provides an axiomatic characterization theorem for both forms of utility functions. For the theorem, we need a definition:

Strong homogeneity: $(\mathbf{x}, \mathbf{y}) \sim (\mathbf{c} * \mathbf{x}, \mathbf{c} * \mathbf{y})$ for all $(\mathbf{x}, \mathbf{y}) \in Y$ and all $\mathbf{c} \gg \mathbf{0}$, where $\mathbf{c} * \mathbf{x} = (c_1 \cdot x_1, c_2 \cdot x_2, \ldots, c_n \cdot x_n) \in X$.
If $c_1 = c_2 = \cdots = c_n$ in the statement of the strong homogeneity axiom, the axiom coincides with the homogeneity axiom, and then the strong homogeneity axiom is stronger than the homogeneity axiom. Moreover, the strong homogeneity axiom implies the strong homotheticity of the level relation, which ensures the existence of a Cobb–Douglas utility function representing the level relation. Formally, we have the following lemma:

Lemma 4. *If a difference relation $\succsim$ satisfies the monotonicity, continuity, consistency and strong homogeneity axioms, then the following assertions hold:*

(i) $\mathbf{x} \sim' \mathbf{y} \Rightarrow \mathbf{c} * \mathbf{x} \sim' \mathbf{c} * \mathbf{y}$ *for all* $\mathbf{c} \gg \mathbf{0}$, *and* $\mathbf{x} \succsim' \mathbf{y} \Rightarrow \mathbf{c} * \mathbf{x} \succsim' \mathbf{c} * \mathbf{y}$ *for all* $\mathbf{c} \gg \mathbf{0}$.
(ii) *There is a function $w : X \to \mathbb{R}$ such that $\mathbf{x} \succsim' \mathbf{y} \Leftrightarrow w(\mathbf{x}) \geq w(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$ and that $w(\mathbf{x}) = x_1^{a_1} \cdot \cdots \cdot x_n^{a_n}$ for some n positive numbers: $a_1 > 0, \ldots, a_n > 0$ satisfying $a_1 + \cdots + a_n = 1$. Moreover, the demand function of $\succsim'$ is given by $d(\mathbf{p}, I) = (a_1 \cdot I/p_1, \ldots, a_n \cdot I/p_n)$.*

The proof of Lemma 4 is given in Appendix B. Specifically, in order to ensure the existence of the Cobb–Douglas utility function Trockel (1989) introduces Condition (i) in Lemma 4, which is called the budget invariance axiom. The condition in Lemma 4 requires the level relations $\succsim'$ to be coordinate-wisely homothetic, that is, the indifference relation $\sim'$ of the level relation is invariant against changes of the units of consumption goods specific to each of the good, while our strong homogeneity axiom above requires the difference relation $\succsim$ to be coordinate-wisely homogeneous, that is, the indifference relation $\sim$ on the differences is invariant against the changes of the units of consumption goods. The main result of this section is the following theorem:

Theorem 3. *Let $\succsim$ be a difference relation on X satisfying the monotonicity, continuity, consistency and quasi-convexity axioms on X. The following statements are mutually equivalent:*

(i) *The difference relation $\succsim$ satisfies the strong homogeneity axiom on X.*
(ii) *There is a utility function of $\succsim$ that is additively separable logarithmic on X.*

[22] The restriction $\Sigma\alpha_i = -\alpha_0$ is needed, because an indirect utility function must be 0-homogeneous on $\mathbb{P} \times \mathbb{I}$.

(iii) *There is an indirect utility function of* $\succsim$ *that is additively separable logarithmic on* $\mathbb{P} \times \mathbb{I}$.

Theorem 3 is proved in Appendix A. Suppose that a difference relation $\succsim$ satisfies all the axioms in Theorem 3, and let w be the Cobb–Douglas function of $\succsim'$ in Lemma 4, that is, $w(\mathbf{x}) = x_1^{a_1} \cdot \cdots \cdot x_n^{a_n}$ ($a_1 > 0, \ldots, a_n > 0$ and $a_1 + \cdots + a_n = 1$). Then it holds by (3) and Theorems 1A and 3 that an extended-real-valued function U on X is a utility function representing $\succsim$ if and only if there exists $\alpha > 0$ and β such that

$$U(\mathbf{x}) = \alpha \cdot (a_1 \cdot \log x_1 + a_2 \cdot \log x_2 + \cdots + a_n \cdot \log x_n) + \beta \quad \text{for all } \mathbf{x} \in X.$$

Moreover, it holds by (5) and Theorems 2 and 3 that an extended-real-valued function V on $\mathbb{P} \times \mathbb{I}$ is an indirect utility function of $\succsim$ if and only if there exists $\alpha > 0$ and β such that

$$\begin{aligned} V(\mathbf{p}, I) &= \alpha \cdot [(-a_1) \cdot \log(p_1/I) + (-a_2) \cdot \log(p_2/I) + \cdots \\ &\quad + (-a_n) \cdot \log(p_n/I)] + \beta \\ &= \alpha \cdot [\log I - (a_1 \cdot \log p_1 + a_2 \cdot \log p_2 + \cdots \\ &\quad + a_n \cdot \log p_n)] + \beta \quad \text{for all } (\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}. \end{aligned}$$

Since $(a_1 \cdot \log p_1 + \cdots + a_n \cdot \log p_n)$ in the equation above is called *Stone's price index*, Theorem 3 implies that the welfare loss (costs) due to the price changes should be measured by the index for the consumers whose preferences satisfy the axioms. Moreover, $V(\mathbf{p}, I)$ can be written as

$$V(\mathbf{p}, I) = \log(I/p_1^{a_1} \cdot \cdots \cdot p_n^{a_n}) \quad \text{for all } (\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}.$$

Hence, if a (representative) consumer satisfies the axioms above, $p_1^{a_1} \cdot \cdots \cdot p_n^{a_n}$ can be recognized as a deflator, because $V(\mathbf{p}, I)$ is a cardinal indirect utility function of the consumer.

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Appendix A

Proof of Theorem 1. (**A**) We can prove (iii) ⇒ (ii) ⇒ (i) easily. We prove (i) ⇒ (ii), and then prove (ii) ⇒ (iii).

[(i) ⇒ (ii)]: Suppose a difference relation $\succsim$ satisfies the four axioms, and let $\succsim'$ be the level relation of $\succsim$. It holds by Lemma 3(iii) that there exists a continuously increasing and 1 homogeneous function $u^* : X \to \mathbb{R}_+$ representing $\succsim'$. Define $U^*(\mathbf{x}) = \log u^*(\mathbf{x})$ for all $\mathbf{x} \in X$. ($U^*(\mathbf{x}) = -\infty$ if $u^*(\mathbf{x}) = 0$. In particular, $U^*(\mathbf{0}) = -\infty$ by $u^*(\mathbf{0}) = 0$.) Then U^* is logarithmically homogeneous on X. We show that U^* is a utility function representing $\succsim$ on X, that is, U^* satisfies (1) in Section 2. Denote $\mathbf{e} \equiv (1, 1, \ldots, 1)$ again, and fix any $(\mathbf{x}, \mathbf{y}), (\mathbf{z}, \mathbf{w}) \in Y$. It holds by Lemma 2(iii, v, vii) that there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+ (\alpha, \gamma > 0)$ such that

$$\mathbf{x} \sim' \alpha \cdot \mathbf{e}, \quad \mathbf{y} \sim' \beta \cdot \mathbf{e}, \quad \mathbf{z} \sim' \gamma \cdot \mathbf{e} \quad \text{and} \quad \mathbf{w} \sim' \delta \cdot \mathbf{e}, \tag{7}$$

which implies that

$$\begin{aligned} &u^*(\mathbf{x}) = u^*(\alpha \cdot \mathbf{e}) = \alpha \cdot u^*(\mathbf{e}), \quad u^*(\mathbf{y}) = \beta \cdot u^*(\mathbf{e}), \\ &u^*(\mathbf{z}) = \gamma \cdot u^*(\mathbf{e}) \quad \text{and} \quad u^*(\mathbf{w}) = \delta \cdot u^*(\mathbf{e}). \end{aligned} \tag{8}$$

It holds by (7), Lemma 2(vi), Lemma 3(ii) and (8) that

$$\begin{aligned} (\mathbf{x}, \mathbf{y}) \succsim (\mathbf{z}, \mathbf{w}) &\Leftrightarrow (\alpha \cdot \mathbf{e}, \beta \cdot \mathbf{e}) \succsim (\gamma \cdot \mathbf{e}, \delta \cdot \mathbf{e}) \Leftrightarrow \beta/\alpha \geq \delta/\gamma \\ &\Leftrightarrow \beta \cdot u^*(\mathbf{e})/\alpha \cdot u^*(\mathbf{e}) \geq \delta \cdot u^*(\mathbf{e})/\gamma \cdot u^*(\mathbf{e}) \\ &\Leftrightarrow u^*(\mathbf{y})/u^*(\mathbf{x}) \geq u^*(\mathbf{w})/u^*(\mathbf{z}) \\ &\Leftrightarrow \log u^*(\mathbf{y}) - \log u^*(\mathbf{x}) \geq \log u^*(\mathbf{w}) - \log u^*(\mathbf{z}) \\ &\Leftrightarrow U^*(\mathbf{y}) - U^*(\mathbf{x}) \geq U^*(\mathbf{w}) - U^*(\mathbf{z}), \end{aligned}$$

which implies that $U^*(\mathbf{x})$ is a utility function representing $\succsim$ on X.

[(ii) ⇒ (iii)]: Suppose there is a logarithmically homogeneous utility function U^* representing $\succsim$, and let U be a utility function representing $\succsim$. Since [(ii) ⇒ (i)] holds, we can assume that $\succsim$ satisfies all the axioms in Assertion (Ai) in Theorem 1 and that $U^*(\mathbf{x}) = \log u^*(\mathbf{x})$, where $u^*(\mathbf{x})$ is a 1-homogeneous function given by Lemma 3(iii). We need a lemma:

Lemma 5. *If a pair of functions* $g : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ *and* $h : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ *satisfies* $g(\beta) - g(\alpha) \geq g(\delta) - g(\gamma) \Leftrightarrow h(\beta) - h(\alpha) \geq h(\delta) - h(\gamma)$ *for all* $\alpha, \beta, \gamma, \delta \in \mathbb{R}_{++}$, *and if there is a positive number* $a^* > 0$ *such that* $h(t) = a^* \cdot \log t$ *for all* $t \in \mathbb{R}_+$, *then there exists* $a > 0$ *and* b *such that* $g(t) = \alpha \cdot h(t) + b$ *for all* $t \in \mathbb{R}_+$.

Setting $g(t) = U(t \cdot \mathbf{e})$ and $h(t) = U^*(t \cdot \mathbf{e}) = [\log u^*(\mathbf{e})] \cdot \log t$ for all $t \geq 0$, it holds by Lemma 5 that there exists $a > 0$ and b such that

$$U(t \cdot \mathbf{e}) = a \cdot U^*(t \cdot \mathbf{e}) + b \quad \text{for all } t \in \mathbb{R}_+. \tag{9}$$

Fix any $\mathbf{x} \in X$. It holds by Lemma 2(v, vii) that there is $\lambda \geq 0$ such that $\lambda \cdot \mathbf{e} \sim' \mathbf{x}$, which implies that $U(\lambda \cdot \mathbf{e}) = U(\mathbf{x})$ and $U^*(\lambda \cdot \mathbf{e}) = U^*(\mathbf{x})$. Thus we have by (9) that $U(\mathbf{x}) = U(\lambda \cdot \mathbf{e}) = a \cdot U^*(\lambda \cdot \mathbf{e}) + b = a \cdot U^*(\mathbf{x}) + b$, which implies that U is logarithmically homogeneous. Hence all the utility functions are logarithmically homogeneous. Next, we prove that the utility function representing $\succsim$ is unique up to the positive affine transformations. If U^0 and U^1 are utility functions representing $\succsim$, then it holds by the same argument above that there exist four numbers, $a^* > 0$, b^*, $c^* > 0$ and d^* such that $U^0(\mathbf{x}) = a^* \cdot U^*(\mathbf{x}) + b^*$ and $U^1(\mathbf{x}) = c^* \cdot U^*(\mathbf{x}) + d^*$ for all $\mathbf{x} \in X$. Setting $\alpha = (a^*/c^*)$ and $\beta = -(a^* \cdot d^*)/c^* + b^*$, we have that

$$U^0(\mathbf{x}) = \alpha \cdot U^1(\mathbf{x}) + \beta \quad \text{for all } \mathbf{x} \in X. \tag{10}$$

Conversely, if a function U^1 represents $\succsim$ and if (10) holds for a function U^0 with two numbers $\alpha > 0$ and β, then U^0 represents $\succsim$.

(**B**) Since $\succsim'$ is strictly quasi-convex, it holds by Friedman (1973) that the 1-homogeneous function $u(\mathbf{x})$ is weakly concave on X. Since the logarithmic function is strongly concave, $U(\mathbf{x}) = \log u(\mathbf{x})$ is strongly concave on X. □

Proof of Theorem 2. We prove (i) ⇒ (ii) ⇒ (i) first, and then prove (ii) ⇒ (iii) ⇒ (ii).

[(i) ⇒ (ii)]: Suppose the difference relation $\succsim$ satisfies the homogeneity axiom. Let u^* be a 1 homogeneous function satisfying all the conditions in Lemma 3(iii). Define a function V^* on $\mathbb{P} \times \mathbb{I}$ by $V^*(\mathbf{p}, I) = \log u^*(d(\mathbf{p}, I))$ for all $(\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}$. Since $\log u^*(\mathbf{x})$ is a logarithmically homogeneous utility function representing $\succsim$ on X, it holds that

$$\begin{aligned} &(d(\mathbf{p}, I), d(\mathbf{q}, J)) \succsim (d(\mathbf{p}^*, I^*), d(\mathbf{q}^*, J^*)) \\ &\quad \Leftrightarrow \log u^*(d(\mathbf{q}, J)) - \log u^*(d(\mathbf{p}, I)) \\ &\qquad\qquad \geq \log u^*(d(\mathbf{q}^*, J^*)) - \log u^*(d(\mathbf{p}^*, I^*)) \\ &\quad \Leftrightarrow V^*(\mathbf{q}, J) - V^*(\mathbf{p}, I) \\ &\qquad\qquad \geq V^*(\mathbf{q}^*, J^*) - V^*(\mathbf{p}^*, I^*) \end{aligned}$$

for all $(\mathbf{p}, I), (\mathbf{q}, J), (\mathbf{p}^*, I^*), (\mathbf{q}^*, J^*) \in \mathbb{P} \times \mathbb{I}$ with $(d(\mathbf{p}, I), d(\mathbf{q}, J)) \in Y$ and $(d(\mathbf{p}^*, I^*), d(\mathbf{q}^*, J^*)) \in Y$. Hence V^* is an indirect utility function of $\succsim$. Next, we show that V^* is logarithmically homogeneous by proving that $v^*(\mathbf{p}) \equiv u^*(d(\mathbf{p}, 1))$ satisfies all the conditions in (5). Since $\succsim'$ is continuous and monotone, it holds by Mas-Colell et al. (1995, Proposition 3.D.3) that $v^*(\mathbf{p})$ is continuously decreasing. We need a lemma:

Lemma 6. *[Well-Known, See* Mas-Colell et al., 1995, *Exercise 3.D.3, page 97, or* Simon and Blume, 1994, *Theorem 22.2]* $d(\lambda \cdot \mathbf{p}, I) = (1/\lambda) \cdot d(\mathbf{p}, I)$ *and* $d(\mathbf{p}, \lambda I) = \lambda \cdot d(\mathbf{p}, I)$ *for all* $(\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}$ *and all* $\lambda > 0$.

It holds by Lemma 6 that $v^*(\mathbf{p})$ is (-1)-homogeneous on $\mathbb{P}$ and that $v^*(+\infty) = 0$, that is,

$$\begin{aligned} v^*(\lambda \cdot \mathbf{p}) &= u^*(d(\lambda \cdot \mathbf{p}, 1)) = u^*((1/\lambda) \cdot d(\mathbf{p}, 1)) \\ &= (1/\lambda) \cdot v^*(\mathbf{p}) \quad \text{for all } \lambda > 0 \text{ and all } \mathbf{p} \in \mathbb{P}, \text{ and} \\ v^*(+\infty) &= \lim_{\lambda \to +\infty} v^*(\lambda \cdot \mathbf{p}) = \lim_{\lambda \to +\infty} (1/\lambda) \cdot v^*(\mathbf{p}) \\ &= \lim_{\lambda \to +\infty} (1/\lambda) \cdot u^*(d(\mathbf{p}, 1)) = 0 \quad \text{for all } \mathbf{p} \in \mathbb{P}. \end{aligned}$$

Moreover, it holds that $V^*(\mathbf{p}, I) = \log u^*[d(\mathbf{p}, I)] = \log u^*[I \cdot d(\mathbf{p}, 1)] = \log u^*[d((1/I)\mathbf{p}, 1)] = 1 \cdot \log v^*(p_1/I, \ldots, p_n/I) + 0$ for all $(\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}$. Thus V^* is logarithmically homogeneous on $\mathbb{P} \times \mathbb{I}$.

[(ii) ⇒ (i)]: Suppose $V(\mathbf{p}, I) = \alpha \cdot \log v(p_1/I, \ldots, p_n/I) + \beta (\alpha > 0)$ is a logarithmically homogeneous indirect utility function of $\succsim$. Fix any $\lambda > 0$ and any $(\mathbf{x}, \mathbf{y}) \in Y$. Since $\succsim'$ is continuous and (strictly) quasi-convex, and since $\mathbf{x} \gg \mathbf{0}$, it holds by Mas-Colell et al. (1995, Separating Hyperplane Theorem, Theorem M.G.3) that there exists $(\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}$ such that $\mathbf{x} = d(\mathbf{p}, I)$. Setting $\mathbf{e} \equiv (1, \ldots, 1)$ again, define a sequence $\{\mathbf{y}^m\}$ by $\mathbf{y}^m = \mathbf{y} + (1/m) \cdot \mathbf{e}$ for all $m = 1, 2, \ldots$, which implies $\lim_m \mathbf{y}^m = \mathbf{y}$ and $\mathbf{y}^m \gg \mathbf{0}$ for all m. Since $\mathbf{y}^m \gg \mathbf{0}$, we can construct a sequence $\{(\mathbf{q}^m, J^m)\}$ such that $d(\mathbf{q}^m, J^m) = \mathbf{y}^m$ for all m. Since $v(q_1^m/J^m, \ldots, q_n^m/J^m)$ and $v(p_1/I, \ldots, p_n/I)$ are (-1)-homogeneous, we have that $V(\mathbf{q}^m, \lambda J^m) - V(\mathbf{p}, \lambda I) = [\alpha \cdot \log v(q_1^m/J^m, \ldots, q_n^m/J^m) + \alpha \cdot \log \lambda] - [\alpha \cdot \log v(p_1/I, \ldots, p_n/I) + \alpha \cdot \log \lambda] = V(\mathbf{q}^m, J^m) - V(\mathbf{p}, I)$, which implies that $(\lambda \cdot \mathbf{x}, \lambda \cdot \mathbf{y}^m) \sim (\mathbf{x}, \mathbf{y}^m)$ for all m. Thus it holds by the continuity axiom that $(\lambda \cdot \mathbf{x}, \lambda \cdot \mathbf{y}) \sim (\mathbf{x}, \mathbf{y})$.

[(ii) ⇒ (iii)]: Let V^* be a logarithmically homogeneous indirect utility function of $\succsim$. Since [(ii) ⇒ (i)] holds, $\succsim$ satisfies the homogeneity axiom. We can assume that $V^*(\mathbf{p}, I) = \log u^*(d(\mathbf{p}, I))$, where u^* is the 1-homogeneous function in Lemma 3(iii). Let V be an indirect utility function of $\succsim$. We show that V is logarithmically homogeneous. Since $\{\mathbf{x} \in X : \mathbf{x} \succsim' \mathbf{e}\}$ is closed and convex, it holds by Mas-Colell et al. (1995, Separating Hyperplane Theorem, Theorem M.G.3) and Lemma 6 that there exists a price vector $\mathbf{p}^* \in \mathbb{P}$ and $t^* > 0$ such that $t^* \cdot d(\mathbf{p}^*, 1) = \mathbf{e}$. Define $\mathbf{p}^0 \in \mathbb{P}$ by $\mathbf{p}^0 = (1/t^*) \cdot \mathbf{p}^*$. Then we have that

$$d(\mathbf{p}^0, 1) = d((1/t^*) \cdot \mathbf{p}^*, 1) = t^* \cdot d(\mathbf{p}^*, 1) = \mathbf{e}.$$

Define two functions g and h on $\mathbb{R}_+$ by $g(t) = V(\mathbf{p}^0, t)$ and $h(t) = V^*(\mathbf{p}^0, t)$, which implies $h(t) = \log u^*(d(\mathbf{p}^0, t)) = \log u^*(t \cdot d(\mathbf{p}^0, 1)) = [\log u^*(\mathbf{e})] \cdot \log t$. Because $g(\beta) - g(\alpha) \geq g(\delta) - g(\gamma) \Leftrightarrow V(\mathbf{p}^0, \beta) - V(\mathbf{p}^0, \alpha) \geq V(\mathbf{p}^0, \delta) - V(\mathbf{p}^0, \gamma) \Leftrightarrow V^*(\mathbf{p}^0, \beta) - V^*(\mathbf{p}^0, \alpha) \geq V^*(\mathbf{p}^0, \delta) - V^*(\mathbf{p}^0, \gamma) \Leftrightarrow h(\beta) - h(\alpha) \geq h(\delta) - h(\gamma)$ for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}_{++}$, it holds by Lemma 5 that there exists $a > 0$ and b such that

$$V(\mathbf{p}^0, I) = a \cdot V^*(\mathbf{p}^0, I) + b \quad \text{for all } I \in \mathbb{I}. \tag{11}$$

Fix any $(\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}$. It holds by Lemma 2(v, vii) that there is $I^0 \in \mathbb{I}$ such that $d(\mathbf{p}^0, I^0) \sim' d(\mathbf{p}, I)$, which implies that $V(\mathbf{p}^0, I^0) = V(\mathbf{p}, I)$ and $V^*(\mathbf{p}^0, I^0) = V^*(\mathbf{p}, I)$. Thus we have by (11) that $V(\mathbf{p}, I) = V(\mathbf{p}^0, I^0) = a \cdot V^*(\mathbf{p}^0, I^0) + b = a \cdot V^*(\mathbf{p}, I) + b$, which implies that V is logarithmically homogeneous. Hence all the indirect utility functions of $\succsim$ are logarithmically homogeneous.

If V^0 and V^1 are indirect utility functions of $\succsim$, then it holds by almost the same arguments in the proof of Theorem 1 that there exists $\alpha > 0$ and β such that

$$V^0(\mathbf{p}, I) = \alpha \cdot V^1(\mathbf{p}, I) + \beta \quad \text{for all } (\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}. \tag{12}$$

Conversely, if V^1 is an indirect utility function of $\succsim$ and if (12) holds for a function V^0 with two numbers $\alpha > 0$ and β, then V^0 is an indirect utility function of $\succsim$.

[(iii) ⇒ (ii)]: Obvious. □

Proof of Theorem 3. It holds obviously that (ii) ⇒ (i). We prove (i) ⇒ (ii), first.

[(i) ⇒ (ii)]: Suppose $\succsim$ satisfies the strong homogeneity axiom. It holds by Lemma 4(ii) that there exists a Cobb–Douglas function representing $\succsim'$, $w(\mathbf{x}) = x_1^{a_1} \cdot \cdots \cdot x_n^{a_n} (a_1 > 0, \ldots, a_n > 0, a_1 + \cdots + a_n = 1)$. Define $U^*(\mathbf{x}) = \log w(\mathbf{x}) = a_1 \cdot \log x_1 + \cdots + a_n \cdot \log x_n$ for all $\mathbf{x} \in X$. Because $w(\mathbf{x})$ satisfies all the conditions in Lemma 3(iii), it holds by Theorem 1(A) that $\succsim$ is represented by U^*.

Second, we prove (i) ⇒ (iii) ⇒ (i).

[(i) ⇒ (iii)]: Suppose $\succsim$ satisfies the strong homogeneity axiom, and let $w(\mathbf{x}) = x_1^{a_1} \cdot \cdots \cdot x_n^{a_n} (a_1 > 0, a_2 > 0, \ldots, a_n > 0, a_1 + \cdots + a_n = 1)$ be the Cobb–Douglas function representing $\succsim'$. Define $V^*(\mathbf{p}, I)$ by $V^*(\mathbf{p}, I) = \log w(d(\mathbf{p}, I)) = \log w(a_1 \cdot I/p_1, \ldots, a_n \cdot I/p_n) = a_1 \cdot \log(a_1 \cdot I/p_1) + \cdots + a_n \cdot \log(a_n \cdot I/p_n)$ for all $(\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}$. Since $w(\mathbf{x})$ satisfies all the conditions in Lemma 3(iii), it holds by Theorem 2 that V^* is an indirect utility function of $\succsim$.

[(iii) ⇒ (i)]: Let $V(\mathbf{p}, I)$ be an indirect utility functions of $\succsim$, and suppose there are $n + 2$ real numbers $\alpha_0 > 0, \alpha_1 < 0, \alpha_2 < 0, \ldots, \alpha_n < 0 (\Sigma \alpha_i = -\alpha_0)$ and β such that

$$\begin{aligned} V(\mathbf{p}, I) = {} & \alpha_0 \cdot \log I + \alpha_1 \cdot \log p_1 + \alpha_2 \cdot \log p_2 + \cdots \\ & + \alpha_n \cdot \log p_n + \beta \quad \text{for all } (\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}. \end{aligned} \tag{13}$$

Fix any $\mathbf{c} \gg \mathbf{0}$ and any $(\mathbf{x}, \mathbf{y}) \in Y$. It holds by Roy's identity that

$$\begin{aligned} & d(\mathbf{p}, I) = (a_1 \cdot I/p_1, \ldots, a_n \cdot I/p_n) \\ & \quad \text{for all } (\mathbf{p}, I) \in \mathbb{P} \times \mathbb{I}, \text{ where } a_i = \alpha_i / \Sigma \alpha_i \text{ for all } i. \end{aligned} \tag{14}$$

Setting $\mathbf{q} = (a_1/x_1, \ldots, a_n/x_n)$ and $\mathbf{r} = (a_1/(c_1 \cdot x_1), \ldots, a_n/(c_n \cdot x_n))$, it holds by (14) that

$$d(\mathbf{q}, 1) = \mathbf{x} \quad \text{and} \quad d(\mathbf{r}, 1) = \mathbf{c} * \mathbf{x}. \tag{15}$$

Define a sequence $\{\mathbf{y}^m\}$ by $\mathbf{y}^m = \mathbf{y} + (1/m) \cdot \mathbf{e}$ for all m, which implies that $\lim \mathbf{y}^m = \mathbf{y}$ and $\mathbf{y}^m \gg \mathbf{0}$ for all m. Define a sequence $\{\mathbf{q}^m\}$ by $\mathbf{q}^m = (a_1/(y_1 + 1/m), \ldots, a_n/(y_n + 1/m))$, and define a sequence $\{\mathbf{r}^m\}$ by $\mathbf{r}^m = (a_1/(c_1 \cdot y_1 + c_1/m), \ldots, a_n/(c_n \cdot y_n + c_n/m))$. It holds by (14) that $d(\mathbf{q}^m, 1) = \mathbf{y}^m \gg \mathbf{0}$ and $d(\mathbf{r}^m, 1) = \mathbf{c} * \mathbf{y}^m \gg \mathbf{0}$ for all m. Hence we have by (13) and (15) that $V(\mathbf{r}^m, 1) - V(\mathbf{r}, 1) = V(\mathbf{q}^m, 1) - V(\mathbf{q}, 1)$ and $(\mathbf{c} * \mathbf{x}, \mathbf{c} * \mathbf{y}^m) \sim (\mathbf{x}, \mathbf{y}^m)$ for all m. Thus it holds by the continuity axiom that $(\mathbf{c} * \mathbf{x}, \mathbf{c} * \mathbf{y}) \sim (\mathbf{x}, \mathbf{y})$. □

Appendix B

Proof of Lemma 1. If a utility function U representing $\succsim$ satisfies (3) for some continuously increasing and 1-homogeneous function $u(\mathbf{x})$, then $\theta = U(e, \ldots, e) - U(1, \ldots, 1) = \log e + U(1, \ldots, 1) - U(1, \ldots, 1) = 1$ and $U(\lambda \cdot \mathbf{x}) = \log u(\lambda \cdot \mathbf{x}) = \log(\lambda \cdot u(\mathbf{x})) = 1 \cdot \log \lambda + \log u(\mathbf{x}) = \theta \cdot \log \lambda + U(\mathbf{x})$, which means that U satisfies (2). Conversely, if a utility function U representing $\succsim$ satisfies (2), then U satisfies (3), by setting $\mu = 0$ and $u(\mathbf{x}) = [e^{U(\mathbf{x})}]^{1/\theta}$ for all $\mathbf{x} \in X$. ($u(\mathbf{0}) = 0$, since $U(\mathbf{0}) = -\infty$). In fact, u is continuously increasing and 1-homogeneous, and it holds that $\theta \cdot \log u(\mathbf{x}) + \mu = \theta \cdot \log[e^{U(\mathbf{x})}]^{1/\theta} + 0 = \theta \cdot (1/\theta) \cdot U(\mathbf{x}) \cdot \log e = U(\mathbf{x})$ for all $\mathbf{x} \in X$. ($\theta \cdot \log u(\mathbf{0}) + \mu = -\infty = U(\mathbf{0})$). □

Proof of Lemma 2. (i) Suppose $(\mathbf{z}, \mathbf{x}) \succ (\mathbf{z}, \mathbf{y})$, and fix any $\mathbf{w} \gg \mathbf{0}$. It holds by the monotonicity axiom that there exists some $\eta > 0$ such that $(\mathbf{z}, \mathbf{y} + \eta \cdot \mathbf{w}) \succ (\mathbf{z}, \mathbf{x}) \succ (\mathbf{z}, \mathbf{y} + 0 \cdot \mathbf{w})$. Since $\succsim$ is continuous, we can prove that $(\mathbf{z}, \mathbf{x}) \sim (\mathbf{z}, \mathbf{y} + \lambda \cdot \mathbf{w})$ for some $\lambda > 0$, using almost the same manner as in Mas-Colell et al. (1995, Figure 3.C.1, p. 47). (ii) Suppose $(\mathbf{z}, \mathbf{x}) \succsim (\mathbf{z}, \mathbf{y})$, and fix any $\mathbf{w} \gg \mathbf{0}$. It holds by Lemma 2(i) that there exists $\lambda \geq 0$ such that $(\mathbf{z}, \mathbf{y} + \lambda \cdot \mathbf{w}) \sim (\mathbf{z}, \mathbf{x})$. Define $\mathbf{y}^* = \mathbf{y} + \lambda \cdot \mathbf{w}$ and define a sequence $\{\mathbf{y}^m\}$ by $\mathbf{y}^m = \mathbf{y}^* + (1/m) \cdot \mathbf{w}$ for all $m = 1, 2, \ldots$. Since $\mathbf{y}^m \gg \mathbf{y}$, it holds by the monotonicity axiom that

$$(\mathbf{z}, \mathbf{y}^m) \succ (\mathbf{z}, \mathbf{y}) \quad \text{and} \quad (\mathbf{w}, \mathbf{y}^m) \succ (\mathbf{w}, \mathbf{y}) \quad \text{for all } m. \tag{16}$$

For each m, it holds by Lemma 2(i) that there exists $t^m > 0$ such that $(\mathbf{z}, \mathbf{x} + t^m \cdot \mathbf{w}) \sim (\mathbf{z}, \mathbf{y}^m)$, since $(\mathbf{z}, \mathbf{y}^m) \succ (\mathbf{z}, \mathbf{y}^*) \sim (\mathbf{z}, \mathbf{x})$. Define a sequence $\{\mathbf{x}^m\}$ by $\mathbf{x}^m = \mathbf{x} + t^m \cdot \mathbf{w}$ for all m. Then it holds that $(\mathbf{z}, \mathbf{x}^m) \sim (\mathbf{z}, \mathbf{y}^m)$ for all m, and it holds by $\mathbf{x}^m, \mathbf{y}^m \gg \mathbf{0}$ and the consistency axiom that

$$(\mathbf{w}, \mathbf{x}^m) \sim (\mathbf{w}, \mathbf{y}^m) \quad \text{for all } m. \tag{17}$$

It holds by (16) and (17) that $(\mathbf{w}, \mathbf{x}^m) \succsim (\mathbf{w}, \mathbf{y})$ for all m. Since $\lim \mathbf{x}^m = \mathbf{x}$, we have by the continuity axiom that $(\mathbf{w}, \mathbf{x}) \succsim (\mathbf{w}, \mathbf{y})$. (iii) Since $\succsim$ is complete and transitive on Y, it holds by Lemma 2(ii) that $\succsim'$ is complete and transitive on X. The monotonicity and continuity axioms imply the monotonicity and continuity of $\succsim'$, respectively. (iv) It holds by (4) and Lemma 2(ii) that $\mathbf{x} \sim' \mathbf{y} \Rightarrow [\mathbf{x} \succsim' \mathbf{y}$ and $\mathbf{y} \succsim' \mathbf{x}] \Rightarrow [(\mathbf{z}, \mathbf{x}) \succsim (\mathbf{z}, \mathbf{y})$ and $(\mathbf{z}, \mathbf{y}) \succsim (\mathbf{z}, \mathbf{x})$ for all $\mathbf{z} \gg \mathbf{0}] \Rightarrow [(\mathbf{z}, \mathbf{x}) \sim (\mathbf{z}, \mathbf{y})$ for all $\mathbf{z} \gg \mathbf{0}]$. Because it holds by (4) that **not** $\mathbf{y} \succsim' \mathbf{x} \Leftrightarrow$ **not** $(\mathbf{z}, \mathbf{y}) \succsim (\mathbf{z}, \mathbf{x})$ for all $\mathbf{z} \gg \mathbf{0}$, it holds by Lemma 2(ii) that $\mathbf{x} \succ' \mathbf{y} \Rightarrow [\mathbf{x} \succsim' \mathbf{y}$ and **not** $\mathbf{y} \succsim' \mathbf{x}] \Rightarrow [(\mathbf{z}, \mathbf{x}) \succsim (\mathbf{z}, \mathbf{y})$ and **not** $(\mathbf{z}, \mathbf{y}) \succsim (\mathbf{z}, \mathbf{x})$ for all $\mathbf{z} \gg \mathbf{0}] \Rightarrow [(\mathbf{z}, \mathbf{x}) \succ (\mathbf{z}, \mathbf{y})$ for all $\mathbf{z} \gg \mathbf{0}]$. (v) Assertion (v) is a direct consequence of Assertion (i). (vi) It holds by Lemma 2(iv) and the consistency axiom that $\mathbf{x}_2 \sim' \mathbf{y}_2 \Rightarrow (\mathbf{x}_1, \mathbf{x}_2) \sim (\mathbf{x}_1, \mathbf{y}_2)$ and that $\mathbf{x}_1 \sim' \mathbf{y}_1 \Rightarrow (\mathbf{y}_2 + (1/m) \cdot \mathbf{x}_1, \mathbf{x}_1) \sim (\mathbf{y}_2 + (1/m) \cdot \mathbf{x}_1, \mathbf{y}_1) \Rightarrow (\mathbf{x}_1, \mathbf{y}_2 + (1/m) \cdot \mathbf{x}_1) \sim (\mathbf{y}_1, \mathbf{y}_2 + (1/m) \cdot \mathbf{x}_1)$ for all $m = 1, 2, \ldots$. Hence we have by the continuity axiom that

$$(\mathbf{x}_1, \mathbf{x}_2) \sim (\mathbf{x}_1, \mathbf{y}_2) \sim (\mathbf{y}_1, \mathbf{y}_2). \tag{18}$$

By almost the same manner, we can prove that $(\mathbf{x}_3, \mathbf{x}_4) \sim (\mathbf{y}_3, \mathbf{y}_4)$. Hence it holds by (18) that $(\mathbf{x}_1, \mathbf{x}_2) \succsim (\mathbf{x}_3, \mathbf{x}_4) \Leftrightarrow (\mathbf{y}_1, \mathbf{y}_2) \succsim (\mathbf{y}_3, \mathbf{y}_4)$. (vii) Suppose $\mathbf{0} \succ' \mathbf{x}$ for some $\mathbf{x} \in X$. Denoting $\mathbf{e} \equiv (1, \ldots, 1)$, it holds by Lemma 2(i) that $(\mathbf{e}, \mathbf{0}) \sim (\mathbf{e}, \mathbf{x} + \lambda \cdot \mathbf{e})$ for some $\lambda > 0$, which is a contradiction. □

Proof of Lemma 3. (i) Suppose $\mathbf{x} \sim' \mathbf{y}$ and $\lambda \cdot \mathbf{x} \succ' \lambda \cdot \mathbf{y}$ for some $\mathbf{x}, \mathbf{y} \in X$ and some $\lambda > 0$. It holds by Lemma 2(iv) that

$$(\mathbf{e}, \mathbf{x}) \sim (\mathbf{e}, \mathbf{y}) \quad \text{and} \quad (\lambda \cdot \mathbf{e}, \lambda \cdot \mathbf{x}) \succ (\lambda \cdot \mathbf{e}, \lambda \cdot \mathbf{y}). \tag{19}$$

It holds by homogeneity axiom that

$$(\mathbf{e}, \mathbf{y}) \sim (\lambda \cdot \mathbf{e}, \lambda \cdot \mathbf{y}) \tag{20}$$

and

$$(\mathbf{e}, \mathbf{x}) \sim (\lambda \cdot \mathbf{e}, \lambda \cdot \mathbf{x}). \tag{21}$$

Hence it holds by (19) and (20) that $(\lambda \cdot \mathbf{e}, \lambda \cdot \mathbf{x}) \succ (\lambda \cdot \mathbf{e}, \lambda \cdot \mathbf{y}) \sim (\mathbf{e}, \mathbf{y}) \sim (\mathbf{e}, \mathbf{x})$, which contradicts with (21). Thus $\mathbf{x} \sim' \mathbf{y} \Rightarrow \lambda \cdot \mathbf{x} \sim' \lambda \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in X$ and all $\lambda > 0$.
(ii) It holds by the monotonicity axiom that

$$\lambda > \mu \Rightarrow (\mathbf{e}, \lambda \cdot \mathbf{e}) \succ (\mathbf{e}, \mu \cdot \mathbf{e}). \tag{22}$$

This implies that $\lambda \geq \mu \Rightarrow (\mathbf{e}, \lambda \cdot \mathbf{e}) \succsim (\mathbf{e}, \mu \cdot \mathbf{e})$. Conversely, suppose $(\mathbf{e}, \lambda \cdot \mathbf{e}) \succsim (\mathbf{e}, \mu \cdot \mathbf{e})$. If $\mu > \lambda$, then it holds by (22) that $(\mathbf{e}, \mu \cdot \mathbf{e}) \succ (\mathbf{e}, \lambda \cdot \mathbf{e})$. This implies **not** $(\mathbf{e}, \lambda \cdot \mathbf{e}) \succsim (\mathbf{e}, \mu \cdot \mathbf{e})$, which is a contradiction. Hence $(\mathbf{e}, \lambda \cdot \mathbf{e}) \succsim (\mathbf{e}, \mu \cdot \mathbf{e}) \Rightarrow \lambda \geq \mu$. It holds by (22) that

$$(\mathbf{e}, \lambda \cdot \mathbf{e}) \succsim (\mathbf{e}, \mu \cdot \mathbf{e}) \Leftrightarrow \lambda \geq \mu \quad \text{for all } \lambda, \mu \geq 0. \tag{23}$$

Next, we prove that $(\alpha \cdot \mathbf{e}, \beta \cdot \mathbf{e}) \succsim (\gamma \cdot \mathbf{e}, \delta \cdot \mathbf{e}) \Leftrightarrow \beta/\alpha \geq \delta/\gamma$ for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$ with $\alpha, \gamma > 0$.
Case 1 ($\beta > 0$ and $\delta > 0$): It holds by the homogeneity axiom that $(\alpha \cdot \mathbf{e}, \beta \cdot \mathbf{e}) \sim (1 \cdot \mathbf{e}, \beta/\alpha \cdot \mathbf{e})$ and $(\gamma \cdot \mathbf{e}, \delta \cdot \mathbf{e}) \sim (1 \cdot \mathbf{e}, \delta/\gamma \cdot \mathbf{e})$, which implies that $(\alpha \cdot \mathbf{e}, \beta \cdot \mathbf{e}) \succsim (\gamma \cdot \mathbf{e}, \delta \cdot \mathbf{e}) \Leftrightarrow (1 \cdot \mathbf{e}, \beta/\alpha \cdot \mathbf{e}) \succsim (1 \cdot \mathbf{e}, \delta/\gamma \cdot \mathbf{e})$. It holds by (23) that $(\alpha \cdot \mathbf{e}, \beta \cdot \mathbf{e}) \succsim (\gamma \cdot \mathbf{e}, \delta \cdot \mathbf{e}) \Leftrightarrow \beta/\alpha \geq \delta/\gamma$.
Case 2 ($\beta = 0$ and $\delta = 0$): It holds that $\beta/\alpha = 0 \geq \delta/\gamma = 0$, and it holds by the homogeneity axiom that $(\alpha \cdot \mathbf{e}, \beta \cdot \mathbf{e}) = (\alpha \cdot \mathbf{e}, 0 \cdot \mathbf{e}) \sim (\gamma \cdot \mathbf{e}, 0 \cdot \gamma/\alpha \cdot \mathbf{e}) = (\gamma \cdot \mathbf{e}, 0 \cdot \mathbf{e}) = (\gamma \cdot \mathbf{e}, \delta \cdot \mathbf{e})$. Hence $(\alpha \cdot \mathbf{e}, \beta \cdot \mathbf{e}) \succsim (\gamma \cdot \mathbf{e}, \delta \cdot \mathbf{e})$.
Case 3 ($\beta > 0$ and $\delta = 0$): It holds that $\beta/\alpha > \delta/\gamma = 0$. It holds by the homogeneity axiom that $(\alpha \cdot \mathbf{e}, \delta \cdot \mathbf{e}) = (\alpha \cdot \mathbf{e}, \mathbf{0}) \sim (\gamma \cdot \mathbf{e}, \mathbf{0}) = (\gamma \cdot \mathbf{e}, \delta \cdot \mathbf{e})$, and it holds by the monotonicity that $(\alpha \cdot \mathbf{e}, \beta \cdot \mathbf{e}) \succ (\gamma \cdot \mathbf{e}, \delta \cdot \mathbf{e})$.
(iii) The existence of a 1-homogeneous utility function $u^* : X \to \mathbb{R}$ representing $\succsim'$ can be proved by Dow and Werlang (1992, Proposition 1.5 and Theorem 1.7). Since it holds by the 1-homogeneity of u^* that $u^*(\mathbf{0}) = 0$, we have $u^*(\mathbf{x}) \geq 0$ by Lemma 2(vii). (iv) Let u be a real-valued function on X. If there exists $a > 0$ such that $u(\mathbf{x}) = a \cdot u^*(\mathbf{x})$ for all $\mathbf{x} \in X$, then v satisfies all the conditions in Lemma 3(iii). Conversely, suppose that u satisfies all the conditions in Lemma 3(iii). Since $u^*(\mathbf{e}) > u^*(\mathbf{0}) = 0$, and since $u(\mathbf{e}) > u(\mathbf{0}) = 0$ by the monotonicity of u, setting $\alpha = u(\mathbf{e})/u^*(\mathbf{e})$, we have that $\alpha > 0$ and

$$u(t, \ldots, t) = t \cdot u(\mathbf{e}) = t \cdot \alpha \cdot u^*(\mathbf{e}) = \alpha \cdot u^*(t, \ldots, t) \quad \text{for all } t \geq 0. \tag{24}$$

Fix any $\mathbf{x} \in X$. It holds by Lemma 2(v, vii) that there is a number $t^0 \geq 0$ such that $(t^0, \ldots, t^0) \sim' \mathbf{x}$, which implies that $u^*(\mathbf{x}) = u^*(t^0, \ldots, t^0)$ and $u(\mathbf{x}) = u(t^0, \ldots, t^0)$. Thus we have by (24) that $u(\mathbf{x}) = u(t^0, \ldots, t^0) = \alpha \cdot u^*(t^0, \ldots, t^0) = \alpha \cdot u^*(\mathbf{x})$. □

Proof of Lemma 4. (i) Suppose $\mathbf{x} \sim' \mathbf{y}$ and $\mathbf{c} * \mathbf{x} \succ' \mathbf{c} * \mathbf{y}$ for some $\mathbf{x}, \mathbf{y} \in X$ and some $\mathbf{c} \gg \mathbf{0}$. It holds by Lemma 2(iv) that

$$(\mathbf{e}, \mathbf{x}) \sim (\mathbf{e}, \mathbf{y}) \quad \text{and} \quad (\mathbf{c} * \mathbf{e}, \mathbf{c} * \mathbf{x}) \succ (\mathbf{c} * \mathbf{e}, \mathbf{c} * \mathbf{y}). \tag{25}$$

It holds by strong homogeneity axiom that

$$(\mathbf{e}, \mathbf{y}) \sim (\mathbf{c} * \mathbf{e}, \mathbf{c} * \mathbf{y}) \tag{26}$$

and

$$(\mathbf{e}, \mathbf{x}) \sim (\mathbf{c} * \mathbf{e}, \mathbf{c} * \mathbf{x}). \tag{27}$$

Hence it holds by (25) and (26) that $(\mathbf{c} * \mathbf{e}, \mathbf{c} * \mathbf{x}) \succ (\mathbf{c} * \mathbf{e}, \mathbf{c} * \mathbf{y}) \sim (\mathbf{e}, \mathbf{y}) \sim (\mathbf{e}, \mathbf{x})$, which contradicts with (27). Thus we have that

$$\mathbf{x} \sim' \mathbf{y} \Rightarrow \mathbf{c} * \mathbf{x} \sim' \mathbf{c} * \mathbf{y} \quad \text{for all } \mathbf{x}, \mathbf{y} \in X \text{ and all } \mathbf{c} \gg \mathbf{0}. \tag{28}$$

Next, we prove the second part. Setting $\mathbf{c}(t) \equiv (t, 1, \ldots, 1)$, it suffices to prove that

$$\mathbf{x} \succ' \mathbf{y} \Rightarrow \mathbf{c}(t) * \mathbf{x} \succsim' \mathbf{c}(t) * \mathbf{y} \quad \text{for all } \mathbf{x}, \mathbf{y} \in X \text{ and all } t > 0. \tag{29}$$

Suppose $\mathbf{x}^0 \succ' \mathbf{y}^0$ and $\mathbf{c}(t^0) * \mathbf{y}^0 \succ' \mathbf{c}(t) * \mathbf{x}^0$ for some $\mathbf{x}^0, \mathbf{y}^0 \in X$ and some $t^0 > 0$.
Case 1($t^0 > 1$): Since $\succsim'$ is monotone and continuous on X, there is a continuous function u on X such that $\mathbf{x} \succsim' \mathbf{y} \Leftrightarrow u(\mathbf{x}) \geq u(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in X$. Define $f(t) = u(\mathbf{c}(t) * \mathbf{x}^0) - u(\mathbf{c}(t) * \mathbf{y}^0)$ for all $t \in [1, t^0]$. Then it holds that $f(1) = u(\mathbf{x}^0) - u(\mathbf{y}^0) > 0$ and $f(t^0) = u(\mathbf{c}(t^0) * \mathbf{x}^0) - u(\mathbf{c}(t^0) * \mathbf{y}^0) < 0$. Since f is continuous on $[1, t^0]$, there exists some $t^* \in [1, t^0]$ such that $f(t^*) = u(\mathbf{c}(t^*) * \mathbf{x}^0) - u(\mathbf{c}(t^*) * \mathbf{y}^0) = 0$, which implies that $\mathbf{c}(t^*) * \mathbf{x}^0 \sim' \mathbf{c}(t^*) * \mathbf{y}^0$. It holds by (28) that $\mathbf{x}^0 = \mathbf{c}(1/t^*) * [\mathbf{c}(t^*) * \mathbf{x}^0] \sim' \mathbf{c}(1/t^*) * [\mathbf{c}(t^*) * \mathbf{y}^0] = \mathbf{y}^0$, which contradicts with $\mathbf{x}^0 \succ' \mathbf{y}^0$. Thus we have that (29) holds.
Case 2($1 > t^0 > 0$): We can prove (29) by almost the same manner as in the proof of Case 1 above.
(ii) It holds by Trockel's (1989) theorem that there exists a Cobb–Douglas utility function representing $\succsim'$. The demand function can be computed easily. □

Proof of Lemma 5. Define a function $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ by $f(t) = g(e^t)$ for all $t \in \mathbb{R}$. Then it holds that

$$f(\log t) = g(e^{\log t}) = g(t) \quad \text{for all } t \in \mathbb{R}_{++}. \tag{30}$$

It holds by the supposition of Lemma 5 that

$$\begin{aligned} \log \beta - \log \alpha = \log \delta - \log \gamma &\Leftrightarrow h(\beta) - h(\alpha) = h(\delta) - h(\gamma) \\ &\Leftrightarrow g(\beta) - g(\alpha) = g(\delta) - g(\gamma) \end{aligned} \tag{31}$$

for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}_{++}$. For all $x, y, z, w \in \mathbb{R}$ set $\alpha = e^x, \beta = e^y, \gamma = e^z$ and $\delta = e^w$. Then we have by (30) and (31) that

$$y - x = w - z \Leftrightarrow f(y) - f(x) = f(w) - f(z) \quad \text{for all } x, y, z, w \in \mathbb{R}. \tag{32}$$

It holds by the supposition of Lemma 5 that $g(\beta) - g(\alpha) \geq g(2) - g(2) \Leftrightarrow h(\beta) - h(\alpha) \geq h(2) - h(2) \Leftrightarrow \log\beta - \log\alpha \geq \log 2 - \log 2$ for all $\alpha, \beta \in \mathbb{R}_{++}$, which implies that $g(\beta) - g(\alpha) \geq 0 \Leftrightarrow \log(\beta/\alpha) \geq \log 1$ and $g(\beta) \geq g(\alpha) \Leftrightarrow \beta \geq \alpha$. Hence g is strictly increasing on $\mathbb{R}_{++}$. Since $e^t > 0$ is strictly increasing on $\mathbb{R}$, $f(t) = g(e^t)$ is strictly increasing on $\mathbb{R}$. It holds by Royden and Fitzpatrick (2010, Section 6.1, Theorem 1) that there are at most countable number of points at which f is not continuous, and then there is a point λ in $\mathbb{R}$ at which f is continuous. Let μ be a point in $\mathbb{R}$, and let $\{\mu^m\}$ be a convergent sequence in $\mathbb{R}$ to μ. Define a sequence $\{\lambda^m\}$ in $\mathbb{R}$ by $\lambda^m = \lambda - \mu + \mu^m$ for all m. Hence we have by (32) that $f(\lambda^m) - f(\lambda) = f(\mu^m) - f(m)$ for all m. Because $\lim \lambda^m = \lambda$ and f is continuous at λ, we have that $\lim f(\mu^m) = f(\mu)$. Thus we have that

$$f(\cdot) \text{ is continuous and increasing on } \mathbb{R}. \tag{33}$$

Setting $a^* = f(1) - f(0) > 0$ and $b^* = f(0)$, we prove that

$$f(t) = a^* \cdot t + b^* \text{ for all rational numbers } t \text{ in } \mathbb{R}. \tag{34}$$

Fix any rational number t in $\mathbb{R}$. There exists a pair of integers (π, θ) such that

$$t = \theta/\pi, \quad \theta \geq 0 \text{ and } \pi \neq 0. \tag{35}$$

Using the induction arguments with respect to $q = 0, 1, 2, \ldots$ for a fixed $p \neq 0$, it holds by (32) that

$$f(q/p) = [f(1/p) - f(0)] \cdot q + f(0) \quad \text{for all integers } q \geq 0 \text{ and } p \neq 0. \tag{36}$$

Case 1 ($t = \theta/\pi \geq 0$): For each $p > 0$, setting $q = p$ in (36), we have that

$$f(1) = [f(1/p) - f(0)] \cdot p + f(0) \quad \text{and}$$
$$f(1/p) = [f(1) - f(0)]/p + f(0) \quad \text{for all } p > 0.$$

It holds by (36) and this that $f(q/p) = (q/p) \cdot [f(1) - f(0)] + f(0)$ for all integers $q \geq 0$ and $p > 0$. Since $a^* = f(1) - f(0)$ and $b^* = f(0)$, we have that $f(t) = f(\theta/\pi) = a^* \cdot (\theta/\pi) + b^* = a^* \cdot t + b^*$.

Case 2 ($t = \theta/\pi < 0$): For each $p < 0$, setting $q = -p > 0$ in (36), we have that

$$f(-1) = -[f(1/p) - f(0)] \cdot p + f(0) \quad \text{and}$$
$$f(1/p) = [f(0) - f(-1)]/p + f(0) \quad \text{for all } p < 0.$$

It holds by (36) and this that $f(q/p) = (q/p) \cdot [f(0) - f(-1)] + f(0)$ for all integers $q > 0$ and $p < 0$. Since $f(1) - f(0) = f(0) - f(-1)$ by (32), we have that $a^* = f(0) - f(-1) > 0$. Hence we have that

$$f(t) = f(\theta/\pi) = a^* \cdot (\theta/\pi) + b^* = a^* \cdot t + b^*.$$

Thus Assertion (34) holds in the both cases.

Because $f(t)$ is continuous on $\mathbb{R}$ by (33), we have by (34) that $f(t) = a^* \cdot t + b^*$ for all real numbers $t \in \mathbb{R}$, which implies that $f(\log t) = a^* \cdot \log t + b^*$. Define the two numbers $a > 0$ and b by $a = a^*/\alpha^*$ and $b = b^*$. Then we have by (30) that

$$g(t) = a \cdot \alpha^* \cdot \log t + b = a \cdot h(t) + b \quad \text{for all } t \in \mathbb{R}_{++}. \tag{37}$$

Since $\lim_{t\to 0} \log t = \log 0 = -\infty$, it holds by (37) that $\lim_{t\to 0} g(t) = g(0) = -\infty$, which implies that

$$g(t) = a \cdot h(t) + b \quad \text{holds for all } t \in \mathbb{R}_+. \quad \square$$

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